

Structured matrices: a deterministic approach for linear systems, nullspace and rank

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Context

And notations

Goal

Design algorithms with **good complexity** bounds for fundamental operations on **polynomials and matrices**

Exact Computations

Base Field $\mathbb{K}$
 $\mathbb{Z}/p\mathbb{Z}, \mathbb{F}_{p^e}, \mathbb{Q}, \dots$

Algebraic Complexity
Upper bound on the
number of operations in $\mathbb{K}$

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Algebraic Complexity
 Upper bound on the number of operations in $\mathbb{K}$

Notations:

- $\tilde{O}$: asymptotic bound hiding logarithmic factors.
- $\mathcal{M}(d)$: cost of multiplying two polynomials of degree less than d in $\mathbb{K}[x]$.
 $\rightsquigarrow O(d \log(d) \log(\log(d))) = \tilde{O}(d)$ on all fields $\mathbb{K}$ [Cantor, Kaltofen 1991].
- ω : exponent of matrix multiplication, $2 < \omega \leq 3$, best known $\omega \approx 2.371$ [Alman et al. 2025].

Structured matrices

Displacement rank approach

Example: Toeplitz matrix

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Example: Toeplitz matrix

$$\mathbb{K}^{4 \times 4}$$

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

Structured matrices

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$$A \in \mathbb{K}^{m \times n}$$

Displacement rank:

The **rank** of the image of A through the **displacement operator** ϕ .

Toeplitz-like structure:

$$\phi : \begin{array}{ccc} \mathbb{K}^{m \times n} & \longrightarrow & \mathbb{K}^{m \times n} \\ A & \longmapsto & A - S_m \cdot A \cdot S_n^T \end{array}$$

$$S_n = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & & 0 \end{bmatrix} \in \mathbb{K}^{n \times n}$$

Structured matrices

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Generators:

$(\mathbf{G}, \mathbf{H}) \in \mathbb{K}^{m \times \alpha} \times \mathbb{K}^{n \times \alpha}$ are called **ϕ -generators of length α for $\mathbf{A}$** if

$$\phi(\mathbf{A}) = \mathbf{G} \cdot \mathbf{H}^T.$$

Structured matrices

Displacement rank approach

Example: Toeplitz matrix

$$\begin{array}{c}
 \mathbb{K}^{4 \times 4} \\
 \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}
 \end{array}
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 \mathbb{S}_m \cdot \mathbf{A} \cdot \mathbb{S}_n^T \\
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 \text{Displacement result} \\
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 \end{array}$$

$$\begin{array}{c}
 \mathbf{G} \\
 \begin{bmatrix} a_0 & 1 \\ a_{-1} & 0 \\ a_{-2} & 0 \\ a_{-3} & 0 \end{bmatrix} \\
 \mathbf{G} \in \mathbb{K}^{4 \times 2}
 \end{array}
 \times
 \begin{array}{c}
 \mathbf{H}^T \\
 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \end{bmatrix} \\
 \mathbf{H} \in \mathbb{K}^{4 \times 2}
 \end{array}
 =
 \begin{array}{c}
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 \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & 0 & 0 & 0 \\ a_{-2} & 0 & 0 & 0 \\ a_{-3} & 0 & 0 & 0 \end{bmatrix} \\
 \text{Displacement rank } \alpha = 2
 \end{array}$$

Displacement operators

Structured-like matrices

$$P \in \mathbb{K}^{m \times m}, Q \in \mathbb{K}^{n \times n}.$$

Stein operator:

$$\Delta[P, Q] : \begin{array}{ccc} \mathbb{K}^{m \times n} & \longrightarrow & \mathbb{K}^{m \times n} \\ A & \longmapsto & A - PAQ \end{array}$$

Sylvester operator:

$$\nabla[P, Q] : \begin{array}{ccc} \mathbb{K}^{m \times n} & \longrightarrow & \mathbb{K}^{m \times n} \\ A & \longmapsto & PA - AQ \end{array}$$

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Choice of P and Q:

$$\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}$$

$\mathbb{D}(x)$

$$\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

S_n

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

S_n^T

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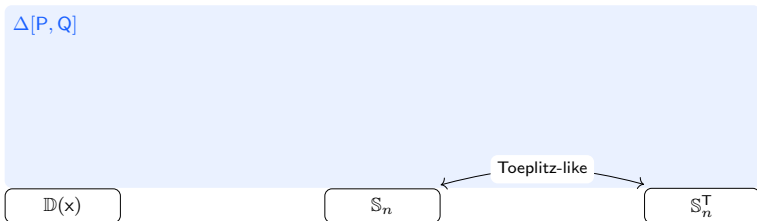
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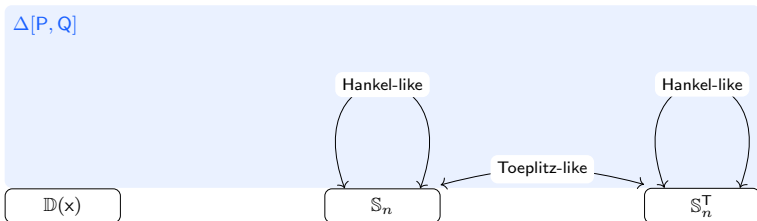
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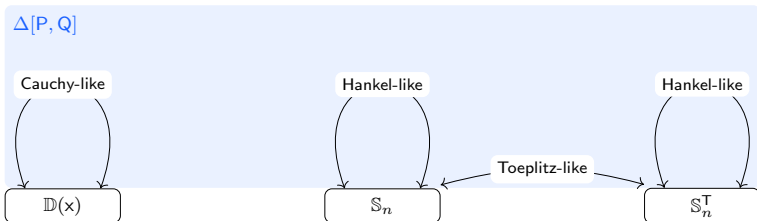
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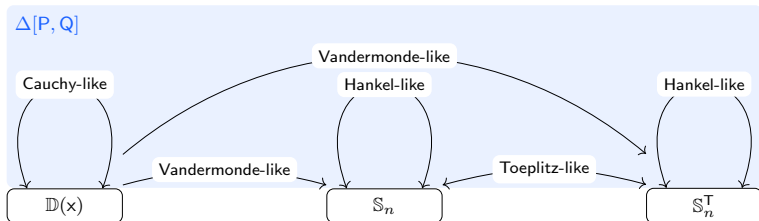
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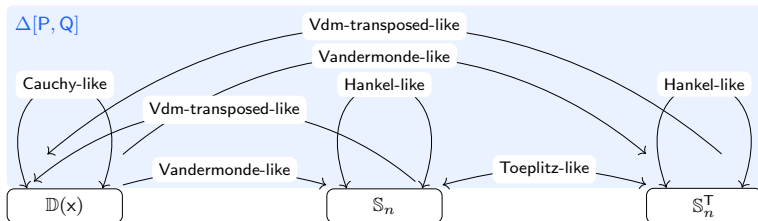
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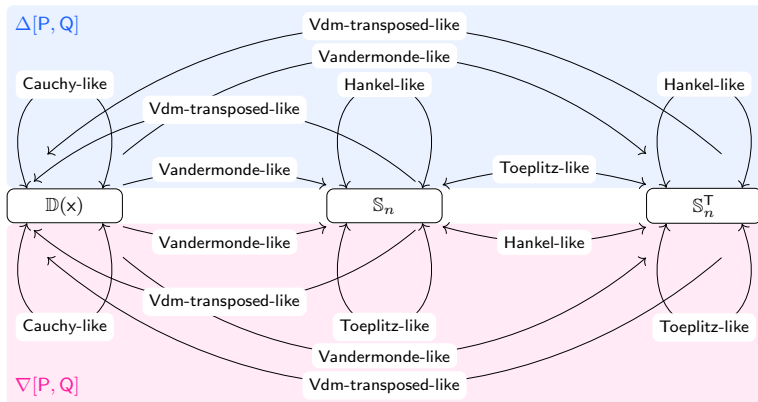
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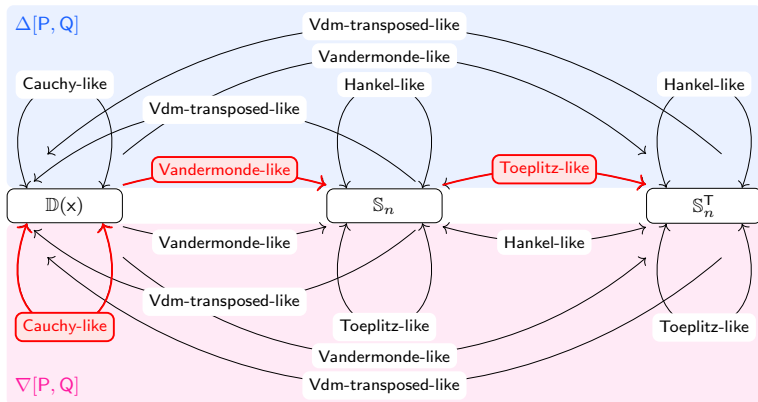
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Choice of P and Q:



Structured matrices

Classical families of structured matrices

Toeplitz matrices ($\alpha = 2$)

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

Toeplitz-like structure:

$$\phi(A) = \Delta[S_m, S_n^T](A) = A - S_m \cdot A \cdot S_n^T$$

Vandermonde matrices ($\alpha = 1$)

$$\begin{bmatrix} 1 & x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{bmatrix}$$

Vandermonde-like structure:

For a repetition-free $x = [x_1, \dots, x_m]$,

$$\phi(A) = \Delta[D(x), S_n^T](A) = A - D(x) \cdot A \cdot S_n^T$$

Cauchy matrices ($\alpha = 1$)

$$\begin{bmatrix} \frac{1}{x_1 - y_1} & \frac{1}{x_1 - y_2} & \frac{1}{x_1 - y_3} \\ \frac{1}{x_2 - y_1} & \frac{1}{x_2 - y_2} & \frac{1}{x_2 - y_3} \\ \frac{1}{x_3 - y_1} & \frac{1}{x_3 - y_2} & \frac{1}{x_3 - y_3} \end{bmatrix}$$

Cauchy-like structure:

For disjoint and repetition-free

$x = [x_1, \dots, x_m], y = [y_1, \dots, y_n]$,

$$\phi(A) = \nabla[D(x), D(y)] = D(x) \cdot A - A \cdot D(y)$$

Structured linear system

Background and motivations

$$A \in \mathbb{K}^{m \times n}$$

Problem 1:

LinearSystem $[\phi, \alpha](G, H, v)$

Using ϕ -generators of A : $G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}, v \in \mathbb{K}^{m \times 1}$

find a nonzero $u \in \mathbb{K}^{n \times 1}$ such that $A \cdot u = v$

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$\rightsquigarrow \phi$ has to be invertible.

Toeplitz-like matrices

$$\phi(A) = A - S_m \cdot A \cdot S_n^T$$

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

Vandermonde-like matrices

$$\phi(A) = A - \mathbb{D}(x) \cdot A \cdot S_n^T$$

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$\rightsquigarrow x$ is repetition-free.

Cauchy-like matrices

$$\phi(A) = \mathbb{D}(x) \cdot A - A \cdot \mathbb{D}(y)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 - y_1 & x_1 - y_2 & x_1 - y_3 \\ 1 & 1 & 1 \\ x_2 - y_1 & x_2 - y_2 & x_2 - y_3 \\ 1 & 1 & 1 \\ x_3 - y_1 & x_3 - y_2 & x_3 - y_3 \end{bmatrix}$$

$\rightsquigarrow x$ and y are disjoint and repetition-free.

Structured linear system

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Problem 2:

Nullspace $[\phi, \alpha](G, H)$

Using ϕ -generators of A: $G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}$
 find $\{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$

Problem 3:

Rank $[\phi, \alpha](G, H)$

Using ϕ -generators of A: $G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}$
 find the rank of A

Structured linear system

Background and motivations

$$\mathbf{A} \in \mathbb{K}^{m \times n}$$

Problem 1:

$$\text{LinearSystem}[\phi, \alpha](\mathbf{G}, \mathbf{H}, \mathbf{v})$$

Using ϕ -generators of $\mathbf{A}$: $\mathbf{G} \in \mathbb{K}^{m \times \alpha}$, $\mathbf{H} \in \mathbb{K}^{n \times \alpha}$, $\mathbf{v} \in \mathbb{K}^{m \times 1}$

find a nonzero $\mathbf{u} \in \mathbb{K}^{n \times 1}$ such that $\mathbf{A} \cdot \mathbf{u} = \mathbf{v}$

State of the art:

$$\bar{m} = \max(m, n)$$

Authors / Year	Structure	Type	Complexity
[Kailath, Kung, Morf 1979]	Toeplitz-like	deterministic	$O(\alpha \bar{m}^2)$
[Bitmead, Anderson 1980], [Morf 1980] <i>Under generic rank profile assumption</i>	Toeplitz-like	genericity	$O(\alpha^2 \mathcal{M}(\bar{m}) \log(\bar{m}))$
[Kaltofen 1994, 1995]	Toeplitz-like	randomized	$O(\alpha^2 \mathcal{M}(\bar{m}) \log(\bar{m}))$
[Pan 1990], [Gohberg, Olshevsky 1994]	Vandermonde-like, Cauchy-like	randomized	Reduction to Toeplitz-like case
[Cardinal 1999], [Pan, Zheng 2000]	Vandermonde-like, Cauchy-like	randomized	$O(\alpha^2 \mathcal{M}(\bar{m}) \log(\bar{m}))$
[Bostan, Jeannerod, Mouilleron, Schost 2008, 2017]	General	randomized	$O(\alpha^{\omega-1} \mathcal{M}(\bar{m}) \log(\bar{m}))$

Structured linear system

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The algorithms are randomized!

Structured linear system

Background and motivations

$$A \in \mathbb{K}^{m \times n}$$

Problem 1:

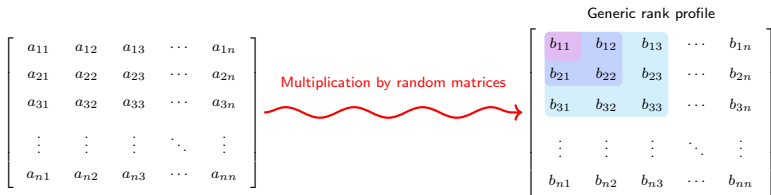
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find a nonzero $u \in \mathbb{K}^{n \times 1}$ such that $A \cdot u = v$

Randomization:

Kaltofen's Leading Principal Inverse algorithm



One of the issues with randomization

The field $\mathbb{K}$ has to be **large enough** for the algorithms to work with positive probability.

E.g., 15000×15000 system with $\alpha = 30$ solved in < 1 sec, but requires $|\mathbb{K}| > 2.25 \cdot 10^8 = 15000^2$.

Structured linear system

Background and motivations

$$A \in \mathbb{K}^{m \times n}$$

Problem 2:

$\text{Nullspace}[\phi, \alpha](G, H)$

Using **generators of A**: find a basis of the nullspace $\{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$

Problem 3:

$\text{Rank}[\phi, \alpha](G, H)$

Using **generators of A**: find the rank of A

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Dense matrix multiplication was not introduced for the structured nullspace and rank problems!

Main result

Fast deterministic algorithm

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find nonzero $u \in \mathbb{K}^{n \times 1}$ such that $A \cdot u = v$

Theorem (K., Neiger 2026)

For $\phi \in \{\text{Toeplitz-like}, \text{Vandermonde-like}, \text{Cauchy-like}\}$ and over any field $\mathbb{K}$:

We can **deterministically** solve LinearSystem $[\phi, \alpha]$ in

$$O(\alpha^{\omega-1}(\mathcal{M}(m) \log(m) + \mathcal{M}(n) \log(n)^2)) = \tilde{O}(\alpha^{\omega-1} \cdot \max(m, n)).$$

Main result

Fast deterministic algorithm

$$A \in \mathbb{K}^{m \times n}$$

Problem 1:

LinearSystem $[\phi, \alpha](G, H, v)$

Using ϕ -generators of A: $G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}, v \in \mathbb{K}^{m \times 1}$

find nonzero $u \in \mathbb{K}^{n \times 1}$ such that $A \cdot u = v$

Theorem (K., Neiger 2026)

For $\phi \in \{\text{Toeplitz-like, Vandermonde-like, Cauchy-like}\}$ and over any field $\mathbb{K}$:

We can deterministically solve LinearSystem $[\phi, \alpha]$ and Nullspace $[\phi, \alpha]$ in

$$O(\alpha^{\omega-1}(\mathcal{M}(m) \log(m) + \mathcal{M}(n) \log(n)^2)) = \tilde{O}(\alpha^{\omega-1} \cdot \max(m, n)).$$

Problem 2:

Nullspace $[\phi, \alpha](G, H)$

Using generators of A:

$$G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}$$

find a basis of the nullspace

$$\{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

Main result

Fast deterministic algorithm

$$A \in \mathbb{K}^{m \times n}$$

Problem 1:

LinearSystem $[\phi, \alpha](G, H, v)$

Using ϕ -generators of A: $G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}, v \in \mathbb{K}^{m \times 1}$

find nonzero $u \in \mathbb{K}^{n \times 1}$ such that $A \cdot u = v$

Theorem (K., Neiger 2026)

For $\phi \in \{\text{Toeplitz-like, Vandermonde-like, Cauchy-like}\}$ and over any field $\mathbb{K}$:

We can deterministically solve LinearSystem $[\phi, \alpha]$, Nullspace $[\phi, \alpha]$ and Rank $[\phi, \alpha]$ in

$$O(\alpha^{\omega-1}(\mathcal{M}(m) \log(m) + \mathcal{M}(n) \log(n)^2)) = \tilde{O}(\alpha^{\omega-1} \cdot \max(m, n)).$$

Problem 2:

Nullspace $[\phi, \alpha](G, H)$

Using generators of A:

$$G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}$$

find a basis of the nullspace

$$\{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

Problem 3:

Rank $[\phi, \alpha](G, H)$

Using generators of A:

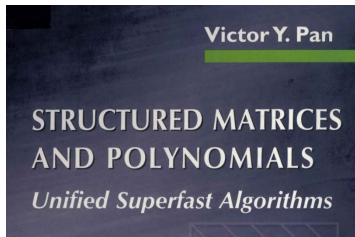
$$G \in \mathbb{K}^{m \times \alpha}, H \in \mathbb{K}^{n \times \alpha}$$

find the rank of A

Key tools: Approximation problems

Structured vs polynomial

Link between the two approaches



$$u(x) \cdot v(x) \bmod x^n$$

Polynomial approach

Polynomial entries

$$\begin{bmatrix} u_1 & 0 & \cdots & 0 \\ u_2 & u_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_n & u_{n-1} & \cdots & u_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Structured approach

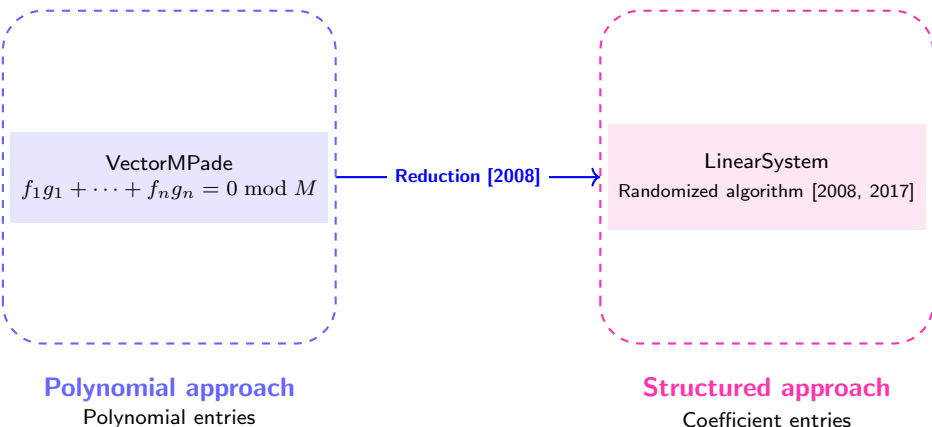
Coefficient entries

Structured vs polynomial

Link between the two approaches

Previous work:

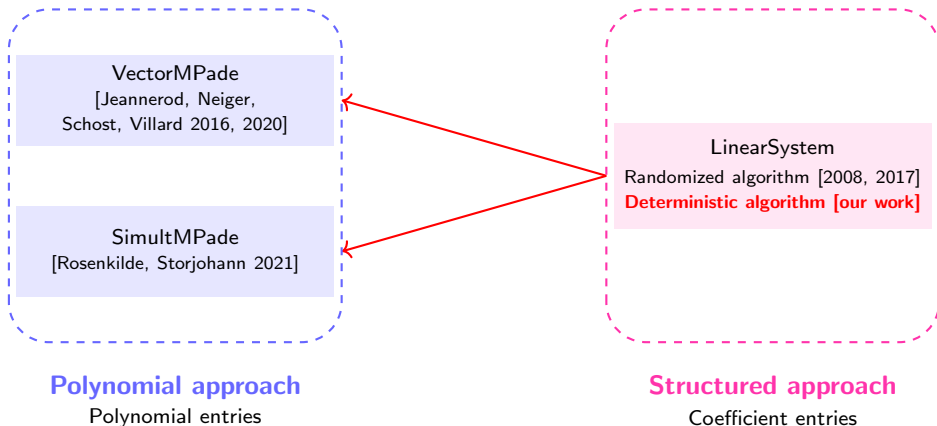
[Bostan, Jeannerod, Mouilleron, Schost 2008, 2017]



Structured vs polynomial

Link between the two approaches

Contribution:



Vector M-Pad  approximation

Problem definition and main results

VectorMPade($M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{1 \times \alpha}, v \in \mathbb{K}[x]$)

Find a small-degree basis of solutions for
 $\{ \mathbf{p} \in \mathbb{K}[x]^\alpha \mid F \cdot \mathbf{p} = v \bmod M \}$

$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

Vector M-Padé approximation

Problem definition and main results

VectorMPade($M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{1 \times \alpha}, v \in \mathbb{K}[x]$)

Find a small-degree basis of solutions for
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$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

Fundamental problems:

- Hermite-Padé approximation when $M = x^n$,
- Berlekamp–Massey algorithm when $M = x^n$ and $\alpha = 2$,
- Rational interpolation when $M = (x - a_1) \cdots (x - a_n)$ with a_i 's distinct.

Vector M-Padé approximation

Problem definition and main results

$\text{VectorMPade}(M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{1 \times \alpha}, v \in \mathbb{K}[x])$

Find a small-degree basis of solutions for
 $\{p \in \mathbb{K}[x]^\alpha \mid F \cdot p = v \bmod M\}$

$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

State of the art:

Authors / Year	Modulus	Type	Complexity
[Beckermann, Labahn 1994] [Giorgi, Jeannerod, Villard 2003]	x^n or $\prod_{i=1}^n (x - x_i)$	homogeneous	$\tilde{O}(\alpha^\omega n)$
[Storjohann 2006] [Zhou, Labahn 2012] [Jeannerod, Neiger, Schost, Villard 2020]	x^n	homogeneous	$O(\alpha^\omega \mathcal{M}(n/\alpha) \log(n))$

Vector M-Pad  approximation

Problem definition and main results

$\text{VectorMPade}(M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{1 \times \alpha}, v \in \mathbb{K}[x])$

Find a small-degree basis of solutions for
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$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

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[Storjohann 2006] [Zhou, Labahn 2012] [Jeannerod, Neiger, Schost, Villard 2020]	x^n	homogeneous	$O(\alpha^\omega \mathcal{M}(n/\alpha) \log(n))$

Nonhomogeneous case ($v \neq 0$) with an arbitrary M .

Vector M-Pad  approximation

Problem definition and main results

VectorMPade($M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{1 \times \alpha}, v \in \mathbb{K}[x]$)

Find a small-degree basis of solutions for
 $\{p \in \mathbb{K}[x]^\alpha \mid F \cdot p = v \bmod M\}$

$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

- **Arbitrary modulus:** Reduction to the case $M = x^n$ in [Neiger 2016].

Nonhomogeneous case ($v \neq 0$) with an arbitrary M .

Vector M-Pad  approximation

Problem definition and main results

VectorMPade($M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{1 \times \alpha}, v \in \mathbb{K}[x]$)

Find a small-degree basis of solutions for
 $\{p \in \mathbb{K}[x]^\alpha \mid F \cdot p = v \bmod M\}$

$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

- **Arbitrary modulus:** Reduction to the case $M = x^n$ in [Neiger 2016].
- **Nonhomogeneous case:**

Input:

$$E \cdot q = 0 \bmod x^n$$

$$E = [F \quad -v] \in \mathbb{K}[x]^{1 \times (\alpha+1)}$$



Output:

$$Q = \begin{bmatrix} P & s \\ 0 & 1 \end{bmatrix} \text{ Popov form}$$

Nonhomogeneous case ($v \neq 0$) with an arbitrary M .

Simultaneous M-Padé approximation

Problem definition and main results

SimultMPade($M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{\alpha \times 1}, v \in \mathbb{K}[x]^\alpha, d = (d_1, \dots, d_\alpha)$)

Find a basis of solutions for

$$\{p \in \mathbb{K}[x], q \in \mathbb{K}[x]^\alpha, \deg(q_i) \leq d_i \mid F \cdot p = (q + v) \bmod M\}$$

$$f_1 p = (q_1 + v_1) \bmod M$$

$$f_2 p = (q_2 + v_2) \bmod M$$

$$\vdots$$

$$f_\alpha p = (q_\alpha + v_\alpha) \bmod M$$

Simultaneous M-Padé approximation

Problem definition and main results

$\text{SimultMPade}(M \in \mathbb{K}[x]_n, F \in \mathbb{K}[x]^{\alpha \times 1}, v \in \mathbb{K}[x]^\alpha, d = (d_1, \dots, d_\alpha))$

Find a basis of solutions for

$$\{p \in \mathbb{K}[x], q \in \mathbb{K}[x]^\alpha, \deg(q_i) \leq d_i \mid F \cdot p = (q + v) \bmod M\}$$

$$f_1 p = (q_1 + v_1) \bmod M$$

$$f_2 p = (q_2 + v_2) \bmod M$$

$$\vdots$$

$$f_\alpha p = (q_\alpha + v_\alpha) \bmod M$$

State of the art:

- [Van Barel, Bultheel 1992] in $O(\alpha^3 n^2)$,
- [Olesh, Storjohann 2007] in $\tilde{O}(\alpha^\omega n)$,
- [Rosenkilde, Storjohann 2016, 2021] in $\underbrace{O(\alpha^\omega \mathcal{M}(n/\alpha) \log(1 + n/\alpha) + \alpha \mathcal{M}(n) \log^2(n))}_{\tilde{O}(\alpha^{\omega-1} n)}$.

$\rightsquigarrow$ computes p only, not the q_i 's.

Structured linear system: Toeplitz-like case

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot \mathbf{u} = \mathbf{v}$

$$A \cdot \mathbf{u} = \mathbf{v}$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot \mathbf{u}(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot u = v$

$$A \cdot u = v$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \cdot \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{ni} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Square Σ -LU decomposition of $A \in \mathbb{K}[x]^{n \times n}$ [Pan 2001]

Given $\phi(A) = G \cdot H^T$, we have $A = \sum_{i=1}^{\alpha} L(G_{*i}) \cdot U(H_{*i})$.

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot u = v$

$$m \leq n$$

$$A \cdot u = v$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \cdot \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{(n-m+1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{(n-m+2)i} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{ni} & \cdots & H_{mi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Σ -LU decomposition of $A \in \mathbb{K}[x]^{m \times n}$ [Pan, Wang 2003]

Given $\phi(A) = G \cdot H^T$, we have $A = \sum_{i=1}^{\alpha} \mathbb{L}_{\min(m,n)}(G_{*i}) \cdot \mathbb{U}_{\min(m,n)}(H_{*i})$.

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot u = v$

$$m \leq n$$

$$A \cdot u = v$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} \text{G}_{1i} & 0 & \cdots & 0 \\ \text{G}_{2i} & \text{G}_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \text{G}_{mi} & \text{G}_{(m-1)i} & \cdots & \text{G}_{1i} \end{bmatrix} \cdot \begin{bmatrix} \text{H}_{ni} & \text{H}_{(n-1)i} & \cdots & \text{H}_{(n-m+1)i} & \cdots & \text{H}_{1i} \\ 0 & \text{H}_{ni} & \cdots & \text{H}_{(n-m+2)i} & \cdots & \text{H}_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \text{H}_{ni} & \cdots & \text{H}_{mi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot u = v$

$$m \leq n$$

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$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

$$h_i(x)u(x) \text{ quo } x^{n-1}$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \cdot \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{(n-m+1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{(n-m+2)i} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{ni} & \cdots & H_{mi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

$$\begin{bmatrix} H_{ni}u_1 + H_{(n-1)i}u_2 + \cdots + H_{(n-m+1)i}u_m + \cdots + H_{1i}u_n \\ H_{ni}u_2 + \cdots + H_{(n-m+2)i}u_m + \cdots + H_{2i}u_n \\ \vdots \\ H_{ni}u_m + \cdots + H_{mi}u_n \end{bmatrix}$$

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot \mathbf{u} = \mathbf{v}$

$$m \leq n$$

$$A \cdot \mathbf{u} = \mathbf{v}$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot \mathbf{u}(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

$$h_i(x)u(x) \text{ quo } x^{n-1}$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \cdot \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{(n-m+1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{(n-m+2)i} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{ni} & \cdots & H_{mi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

$$g_i(x) \cdot (h_i(x)u(x) \text{ quo } x^{n-1}) \bmod x^m$$

$$\begin{bmatrix} G_{1i}p_1 \\ G_{2i}p_1 + G_{1i}p_2 \\ \vdots \\ G_{mi}p_1 + G_{(m-1)i}p_2 + \cdots + G_{1i}p_m \end{bmatrix}$$

$$\begin{bmatrix} H_{ni}u_1 + H_{(n-1)i}u_2 + \cdots + H_{(n-m+1)i}u_m + \cdots + H_{1i}u_n \\ H_{ni}u_2 + \cdots + H_{(n-m+2)i}u_m + \cdots + H_{2i}u_n \\ \vdots \\ H_{ni}u_m + \cdots + H_{mi}u_n \end{bmatrix}$$

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot u = v$

$$m \leq n$$

$$A \cdot u = v$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

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LinearSystem[Toeplitz-like, α](G, H, v)

Find u such that

$$A \cdot u = v$$

Polynomial
formulation

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

Vector M-Pad  approximation

First step

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

$$\Leftrightarrow [g_1(x) \quad \cdots \quad g_{\alpha}(x)] \cdot \left(\begin{bmatrix} h_1(x) \\ \vdots \\ h_{\alpha}(x) \end{bmatrix} \cdot u(x) \text{ quo } x^{n-1} \right) = v(x) \bmod x^m$$

Vector M-Pad  approximation

First step

$$\text{Find } u(x) \text{ such that} \\ \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{F \cdot p = v(x) \bmod x^m\}$

$$\Leftrightarrow \underbrace{\begin{bmatrix} g_1(x) & \cdots & g_{\alpha}(x) \end{bmatrix}}_{F \in \mathbb{K}[x]^{1 \times \alpha}} \cdot \begin{bmatrix} p_1(x) \\ \vdots \\ p_{\alpha}(x) \end{bmatrix} = v(x) \bmod x^m$$

Vector M-Padé approximation

First step

$$\text{Find } u(x) \text{ such that } \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{F \cdot p = v(x) \bmod x^m\}$

$$\Leftrightarrow \underbrace{\begin{bmatrix} g_1(x) & \cdots & g_{\alpha}(x) \end{bmatrix}}_{F \in \mathbb{K}[x]^{1 \times \alpha}} \cdot \begin{bmatrix} p_1(x) \\ \vdots \\ p_{\alpha}(x) \end{bmatrix} = v(x) \bmod x^m$$

- $P \in \mathbb{K}[x]^{\alpha \times \alpha}$ a Popov basis of $\{q \in \mathbb{K}[x]^{\alpha \times 1} \mid F \cdot q = 0 \bmod x^m\}$
- $s \in \mathbb{K}[x]^{\alpha \times 1}$ such that $F \cdot s = v(x) \bmod x^m$

$$\begin{bmatrix} p_1(x) \\ \vdots \\ p_{\alpha}(x) \end{bmatrix} = P \cdot \lambda + s$$

Vector M-Padé approximation

First step

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{F \cdot p = v(x) \bmod x^m\}$

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- $P \in \mathbb{K}[x]^{\alpha \times \alpha}$ a Popov basis of $\{q \in \mathbb{K}[x]^{\alpha \times 1} \mid F \cdot q = 0 \bmod x^m\}$
- $s \in \mathbb{K}[x]^{\alpha \times 1}$ such that $F \cdot s = v(x) \bmod x^m$

$$\begin{bmatrix} h_1(x) \\ \vdots \\ h_{\alpha}(x) \end{bmatrix} u(x) \text{ quo } x^{n-1} = P \cdot \lambda + s$$

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

VectorMPade

Find $u(x)$ such that

$$h \cdot u(x) \text{ quo } x^{n-1} = P \cdot \lambda + s$$

Vector M-Padé approximation

First step

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{F \cdot p = v(x) \bmod x^m\}$

$$\Leftrightarrow \underbrace{\begin{bmatrix} g_1(x) & \cdots & g_{\alpha}(x) \end{bmatrix}}_{F \in \mathbb{K}[x]^{1 \times \alpha}} \cdot \begin{bmatrix} p_1(x) \\ \vdots \\ p_{\alpha}(x) \end{bmatrix} = v(x) \bmod x^m$$

- $P \in \mathbb{K}[x]^{\alpha \times \alpha}$ a Popov basis of $\{q \in \mathbb{K}[x]^{\alpha \times 1} \mid F \cdot q = 0 \bmod x^m\}$
- $s \in \mathbb{K}[x]^{\alpha \times 1}$ such that $F \cdot s = v(x) \bmod x^m$

$$\begin{bmatrix} \text{rev}_{n-1}(h_1(x)) \\ \vdots \\ \text{rev}_{n-1}(h_{\alpha}(x)) \end{bmatrix} \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) = v(x) \bmod x^m$$

VectorMPade

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

$$\Leftrightarrow \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{s} - \bar{P} \cdot \bar{\lambda} = 0 \bmod x^n$$

$$\Leftrightarrow \begin{bmatrix} \bar{h} & -\bar{s} & -\bar{P} \end{bmatrix} \cdot \begin{bmatrix} \text{rev}_{n-1}(u(x)) \\ 1 \\ \bar{\lambda} \end{bmatrix} = 0 \bmod x^n$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

$$\Leftrightarrow \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{s} - \bar{P} \cdot \bar{\lambda} = 0 \bmod x^n$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} \bar{h} & -\bar{s} & -\bar{P} \end{bmatrix}}_{F \in \mathbb{K}[x]^{\alpha \times (\alpha+2)}} \cdot \begin{bmatrix} \text{rev}_{n-1}(u(x)) \\ 1 \\ \bar{\lambda} \end{bmatrix} = 0 \bmod x^n$$

Compressing the problem

Second step

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$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

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$$\Leftrightarrow \bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \cdot \bar{s} = \bar{\lambda} \bmod x^n$$

Compressing the problem

Second step

$$\text{Find } u(x) \text{ such that}$$

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

$$\Leftrightarrow \bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \cdot \bar{s} = \bar{\lambda} \bmod x^n$$

Definition

P is in Popov form $\Leftrightarrow \text{clm}(P)$ is invertible and $\text{rlm}(P) = \mathbb{I}_\alpha$.

Example.

$$P = \begin{bmatrix} \boxed{3 + 2x + 0x^2} & \boxed{1 + 1x^3} \\ \boxed{1 + 1x^2} & \boxed{1 + 4x + 0x^3} \end{bmatrix}$$

degree 2 degree 3

$$\text{clm}(P) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ invertible}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

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Definition

P is in Popov form $\Leftrightarrow \text{clm}(P)$ is invertible and $\text{rlm}(P) = \mathbb{I}_\alpha$.

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$$P = \left[\begin{array}{c|c} \begin{array}{c} 3 + 2x + 0x^2 \\ 1 + 1x^2 \end{array} & \begin{array}{c} 1 + 1x^3 \\ 1 + 4x + 0x^3 \end{array} \end{array} \right]$$

degree 2 degree 3

$$\text{clm}(P) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ invertible}$$

$$\bar{P} = \left[\begin{array}{c|c} \begin{array}{c} 3x^3 + 2x + 0 \\ x^2 + 1 \end{array} & \begin{array}{c} x^3 + 1 \\ x^3 + 4x^2 + 0 \end{array} \end{array} \right]$$

$$\bar{P}(0) \text{ is invertible} \Rightarrow \det(\bar{P})(0) \neq 0 \Rightarrow$$

$\det(\bar{P})$ is coprime with x .

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

$$\Leftrightarrow \bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \cdot \bar{s} = \bar{\lambda} \bmod x^n$$

$\bar{P}$ invertible mod x^n :

$$\bar{P}^{-1} \bar{h} \bmod x^n$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

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$\bar{P}$ invertible mod x^n :

$$\bar{P}^{-1} \bar{h} \bmod x^n$$

K basis of the right kernel

$$\{k \in \mathbb{K}[x]^{\alpha+1} \mid [\bar{P} \quad -\bar{h}] \cdot k = 0\}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

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K basis of the right kernel

$$\{k \in \mathbb{K}[x]^{\alpha+1} \mid \underbrace{\begin{bmatrix} \bar{P} & -\bar{h} \end{bmatrix}}_{\text{full rank}} \cdot k = 0\}$$

$$K = \begin{bmatrix} u \\ \mu \end{bmatrix}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

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K basis of the right kernel

$$\{k \in \mathbb{K}[x]^{\alpha+1} \mid \underbrace{\begin{bmatrix} \bar{P} & -\bar{h} \end{bmatrix}}_{\text{full rank}} \cdot k = 0\}$$

$$K = \begin{bmatrix} u \\ \mu \end{bmatrix}$$

$$\mu^{-1} u \bmod x^n$$

Compressing the problem

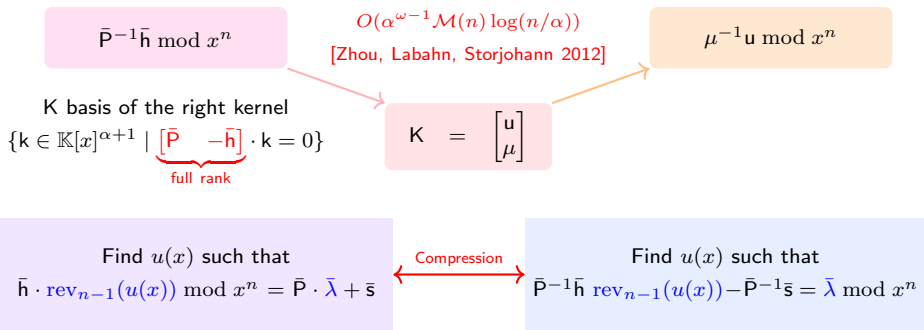
Second step

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$\bar{P}$ invertible mod x^n :



Simultaneous M-Padé approximation

Third step

Find $u(x)$ such that

$$\bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \cdot \bar{s} = \bar{\lambda} \bmod x^n$$

$$\Leftrightarrow \bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) = (\bar{P}^{-1} \cdot \bar{s} + \bar{\lambda}) \bmod x^n$$

Simultaneous M-Padé approximation

Third step

$$\text{Find } u(x) \text{ such that} \\ \bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \cdot \bar{s} = \bar{\lambda} \bmod x^n$$

$$\Leftrightarrow \underbrace{\bar{P}^{-1} \cdot \bar{h}}_{F \in \mathbb{K}[x]^{\alpha \times 1}} \cdot \text{rev}_{n-1}(u(x)) = \underbrace{(\bar{P}^{-1} \cdot \bar{s} + \bar{\lambda})}_{v \in \mathbb{K}[x]^\alpha} \bmod x^n$$

$$f_1(x) \cdot \text{rev}_{n-1}(u(x)) = (v_1(x) + \bar{\lambda}_1(x)) \bmod x^n$$

$$f_2(x) \cdot \text{rev}_{n-1}(u(x)) = (v_2(x) + \bar{\lambda}_2(x)) \bmod x^n$$

$$\vdots$$

$$f_\alpha(x) \cdot \text{rev}_{n-1}(u(x)) = (v_\alpha(x) + \bar{\lambda}_\alpha(x)) \bmod x^n$$

Simultaneous M-Padé approximation

Third step

$$\text{Find } u(x) \text{ such that} \\ \bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \cdot \bar{s} = \bar{\lambda} \bmod x^n$$

$$\Leftrightarrow \underbrace{\bar{P}^{-1} \cdot \bar{h}}_{F \in \mathbb{K}[x]^{\alpha \times 1}} \cdot \text{rev}_{n-1}(u(x)) = \underbrace{(\bar{P}^{-1} \cdot \bar{s} + \bar{\lambda})}_{v \in \mathbb{K}[x]^{\alpha}} \bmod x^n$$

$$f_1(x) \cdot \text{rev}_{n-1}(u(x)) = (v_1(x) + \bar{\lambda}_1(x)) \bmod x^n$$

$$f_2(x) \cdot \text{rev}_{n-1}(u(x)) = (v_2(x) + \bar{\lambda}_2(x)) \bmod x^n$$

$$\vdots$$

$$f_{\alpha}(x) \cdot \text{rev}_{n-1}(u(x)) = (v_{\alpha}(x) + \bar{\lambda}_{\alpha}(x)) \bmod x^n$$

$$P = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{array}{l} \text{degree } \delta_1 \\ \text{degree } \delta_2 \\ \\ \text{degree } \delta_{\alpha} \end{array}$$

with respect to the degrees $d = (n - \delta_1, \dots, n - \delta_2, n - \delta_{\alpha})$.

SimultMPade: Find a partial basis of solutions for $\{F \cdot \text{rev}_{n-1}(u(x)) = (v + \bar{\lambda}) \bmod x^n\}$

Simultaneous M-Padé approximation

Third step

$$\text{Find } u(x) \text{ such that} \\ \bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \cdot \bar{s} = \bar{\lambda} \bmod x^n$$

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$$f_1(x) \cdot \text{rev}_{n-1}(u(x)) = (v_1(x) + \bar{\lambda}_1(x)) \bmod x^n$$

$$f_2(x) \cdot \text{rev}_{n-1}(u(x)) = (v_2(x) + \bar{\lambda}_2(x)) \bmod x^n$$

$$\vdots$$

$$f_\alpha(x) \cdot \text{rev}_{n-1}(u(x)) = (v_\alpha(x) + \bar{\lambda}_\alpha(x)) \bmod x^n$$

$$P = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

degree δ_1
degree δ_2

degree δ_α

with respect to the degrees $d = (n - \delta_1, \dots, n - \delta_2, n - \delta_\alpha)$.

SimultMPade: Find a partial basis of solutions for $\{F \cdot \text{rev}_{n-1}(u(x)) = (v + \bar{\lambda}) \bmod x^n\}$

$$\text{Find } u(x) \text{ such that} \\ \bar{P}^{-1} \bar{h} \text{rev}_{n-1}(u(x)) - \bar{P}^{-1} \bar{s} = \bar{\lambda} \bmod x^n$$

SimultMPade $\rightarrow$

Output $\text{rev}_{n-1}(u(x))$

Toeplitz-like linear system

Overview of the deterministic algorithm

LinearSystem[**Toeplitz-like**, α](G, H, v)

Using generators of A, find u such that $A \cdot u = v$

$$\sum_{i=1}^{\alpha} L_m(G_{*i}) \cdot U_m(H_{*i}) \cdot u = v$$

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot u(x) \text{ quo } x^{n-1}) \bmod x^m = v(x)$$

$$\text{VectorMPade } \tilde{O}(\alpha^{\omega-1} \cdot m)$$

[Jeannerod, Neiger, Villard 2020]

Find $u(x)$ such that

$$\bar{h} \cdot \text{rev}_{n-1}(u(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda} + \bar{s}$$

$$\text{Compression } \tilde{O}(\alpha^{\omega-1} \cdot n)$$

[Zhou, Labahn, Storjohann 2012]

Find $u(x)$ such that

$$\bar{P}^{-1} \cdot \bar{h} \cdot \text{rev}_{n-1}(u(x)) = (\bar{P}^{-1} \cdot \bar{s} + \bar{\lambda}) \bmod x^n$$

$$\text{SimultMPade } \tilde{O}(\alpha^{\omega-1} \cdot n)$$

[Rosenkilde, Storjohann 2021]

Output $\text{rev}_{n-1}(u(x))$

$\tilde{O}(\alpha^{\omega-1} \cdot n)$
Deterministic

Nullspace and rank

Nullspace

The case where $v = 0$

$$A \in \mathbb{K}^{m \times n}$$

Problem 2:

Nullspace[Toeplitz-like, α](G, H)

$$\text{Find } \{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

Nullspace

The case where $v = 0$

$$A \in \mathbb{K}^{m \times n}$$

Problem 2:

$$\text{Nullspace}[\text{Toeplitz-like}, \alpha](G, H)$$

$$\text{Find } \{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

Step zero:

$$\text{Nullspace}[\text{Toeplitz-like}, \alpha](G, H)$$

$$\text{Find } \{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

Polynomial
formulation

$$\text{Find } z(x) \text{ such that}$$

$$\sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot z(x) \bmod x^{n-1}) = 0 \bmod x^m$$

$$\begin{matrix} g_i(x) \\ \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \end{matrix} \cdot \begin{matrix} h_i(x) \\ \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{(n-m+1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{(n-m+2)i} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{ni} & \cdots & H_{mi} \end{bmatrix} \end{matrix} \cdot \begin{matrix} u(x) \\ \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \end{matrix} = \begin{matrix} 0 \\ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{matrix}$$

Nullspace

The case where $v = 0$

$$A \in \mathbb{K}^{m \times n}$$

Problem 2:

$$\text{Nullspace}[\text{Toeplitz-like}, \alpha](G, H) \\ \text{Find } \{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

First and second steps:

- $P \in \mathbb{K}[x]^{\alpha \times \alpha}$ a Popov basis of $\{q \in \mathbb{K}[x]^{\alpha \times 1} \mid F \cdot q = 0 \bmod x^m\}$
- No specific solution, i.e. $s = 0$.

$$\text{Find } z(x) \text{ such that} \\ \sum_{i=1}^{\alpha} g_i(x) \cdot (h_i(x) \cdot z(x) \text{ quo } x^{n-1}) \\ = 0 \bmod x^m$$

VectorMPade

$$\text{Find } z(x) \text{ such that} \\ \bar{h} \cdot \text{rev}_{n-1}(z(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda}$$

$$\text{Find } z(x) \text{ such that} \\ \bar{h} \cdot \text{rev}_{n-1}(z(x)) \bmod x^n = \bar{P} \cdot \bar{\lambda}$$

Compression

$$\text{Find } z(x) \text{ such that} \\ \bar{P}^{-1} \bar{h} \text{rev}_{n-1}(z(x)) = \bar{\lambda} \bmod x^n$$

Nullspace

The case where $v = 0$

$$A \in \mathbb{K}^{m \times n}$$

Problem 2:

$$\text{Nullspace}[\text{Toeplitz-like}, \alpha](G, H)$$

$$\text{Find } \{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

Third step:

$$\text{Find } z(x) \text{ such that } \bar{P}^{-1} \bar{h} \text{ rev}_{n-1}(z(x)) = \bar{\lambda} \bmod x^n \xrightarrow{\text{SimultMPade}} \text{Output } \text{rev}_{n-1}(z(x))$$

Output: a partial basis with respect to the degrees $d = (n - \delta_1, \dots, n - \delta_2, n - \delta_\alpha)$.

$$P = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{matrix} \text{degree } \delta_1 \\ \text{degree } \delta_2 \\ \\ \text{degree } \delta_\alpha \end{matrix}$$

$$\begin{bmatrix} \lambda_1(x) \\ \lambda_2(x) \\ \vdots \\ \lambda_\alpha(x) \end{bmatrix} \begin{matrix} < \text{degree } n - \delta_1 \\ < \text{degree } n - \delta_2 \\ \\ < \text{degree } n - \delta_\alpha \end{matrix}$$

Nullspace

The case where $v = 0$

$$A \in \mathbb{K}^{m \times n}$$

Problem 2:

$$\text{Nullspace}[\text{Toeplitz-like}, \alpha](G, H) \\ \text{Find } \{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$$

Third step:

$$\text{Find } z(x) \text{ such that } \bar{P}^{-1} \bar{h} \text{ rev}_{n-1}(z(x)) = \bar{\lambda} \bmod x^n \xrightarrow{\text{SimultMPade}} \text{Output } \text{rev}_{n-1}(z(x))$$

Output: a partial basis with respect to the degrees $d = (n - \delta_1, \dots, n - \delta_2, n - \delta_\alpha)$.

$$P = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{array}{l} \text{degree } \delta_1 \\ \text{degree } \delta_2 \\ \\ \text{degree } \delta_\alpha \end{array} \quad \begin{bmatrix} \lambda_1(x) \\ \lambda_2(x) \\ \vdots \\ \lambda_\alpha(x) \end{bmatrix} \begin{array}{l} < \text{degree } n - \delta_1 \\ < \text{degree } n - \delta_2 \\ \\ < \text{degree } n - \delta_\alpha \end{array}$$

The nullspace of A is generated by the set $\left\{ \sum_{i=1}^{\ell} x^{n-d_i-t_i} p_i q_i \mid q_i \in \mathbb{K}[x]_{< t_i} \text{ for } 1 \leq i \leq \ell \right\}$.

Displacement generators of the nullspace

Structure of the nullspace basis

$$A \in \mathbb{K}^{m \times n}$$

The nullspace of A is generated by the set

$$\left\{ \sum_{i=1}^{\ell} x^{n-d_i-t_i} p_i q_i \mid q_i \in \mathbb{K}[x]_{< t_i} \text{ for } 1 \leq i \leq \ell \right\}$$



$$\left\{ N\gamma \in \mathbb{K}^{n \times 1} \mid \gamma \in \mathbb{K}^{\sum_{i=1}^{\ell} t_i} \right\}$$

Displacement generators of the nullspace

Structure of the nullspace basis

$$A \in \mathbb{K}^{m \times n}$$

The nullspace of A is generated by the set

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$$\left\{ N\gamma \in \mathbb{K}^{n \times 1} \mid \gamma \in \mathbb{K}^{\sum_{i=1}^{\ell} t_i} \right\}$$

$$\gamma = \begin{bmatrix} q_{11} \\ \vdots \\ q_{1t_1} \\ \vdots \\ q_{\ell 1} \\ \vdots \\ q_{\ell t_\ell} \end{bmatrix}$$

$$N = \begin{bmatrix} N_1 & \cdots & N_\ell \end{bmatrix} = \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ p_{12} & p_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{1d_1} & p_{1d_1-1} & \cdots & p_{11} \end{bmatrix} \cdots \begin{bmatrix} p_{\ell 1} & 0 & \cdots & 0 \\ p_{\ell 2} & p_{\ell 1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{\ell d_\ell} & p_{\ell d_\ell-1} & \cdots & p_{\ell 1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Displacement generators of the nullspace

Structure of the nullspace basis

$$A \in \mathbb{K}^{m \times n}$$

The nullspace of A is generated by the set

$$\left\{ \sum_{i=1}^{\ell} x^{n-d_i-t_i} p_i q_i \mid q_i \in \mathbb{K}[x]_{< t_i} \text{ for } 1 \leq i \leq \ell \right\}$$



$$\left\{ N\gamma \in \mathbb{K}^{n \times 1} \mid \gamma \in \mathbb{K}^{\sum_{i=1}^{\ell} t_i} \right\}$$

$$\gamma = \begin{bmatrix} q_{11} \\ \vdots \\ q_{1t_1} \\ \vdots \\ q_{\ell 1} \\ \vdots \\ q_{\ell t_\ell} \end{bmatrix}$$

$$N = [N_1 \quad \cdots \quad N_\ell] = \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ p_{12} & p_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{1d_1} & p_{1d_1-1} & \cdots & p_{11} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdots \begin{bmatrix} p_{\ell 1} & 0 & \cdots & 0 \\ p_{\ell 2} & p_{\ell 1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{\ell d_\ell} & p_{\ell d_\ell-1} & \cdots & p_{\ell 1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Lemma

N has a Toeplitz-like structure (displacement rank ℓ), i.e., $\Delta[S_n, S_{\sum_{i=1}^{\ell} t_i}](N) = GH^T$

Displacement generators of the nullspace

Structure of the nullspace basis

$$A \in \mathbb{K}^{m \times n}$$

The nullspace of A is generated by the set

$$\left\{ \sum_{i=1}^{\ell} x^{n-d_i-t_i} p_i q_i \mid q_i \in \mathbb{K}[x]_{< t_i} \text{ for } 1 \leq i \leq \ell \right\}$$



$$\left\{ \mathbf{N}\gamma \in \mathbb{K}^{n \times 1} \mid \gamma \in \mathbb{K}^{\sum_{i=1}^{\ell} t_i} \right\}$$

$$\gamma = \left\{ \begin{bmatrix} q_{11} \\ \vdots \\ q_{1t_1} \\ \vdots \\ q_{\ell 1} \\ \vdots \\ q_{\ell t_\ell} \end{bmatrix} \right\} \sum_{i=1}^{\ell} t_i \quad \mathbf{N} = \left[\begin{array}{ccc} \underbrace{\sum_{i=1}^{\ell} t_i}_{\substack{\ell \\ \left\{ \begin{array}{c} \text{pink box } G \end{array} \right\}}} & \cdots & \underbrace{\sum_{i=1}^{\ell} t_i}_{\substack{\ell \\ \left\{ \begin{array}{c} \text{blue box } H^T \end{array} \right\}}} \end{array} \right] = \left[\begin{array}{cccc} p_{11} & 0 & \cdots & 0 \\ p_{12} & p_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{1d_1} & p_{1d_1-1} & \cdots & p_{11} \end{array} \right] \cdots \left[\begin{array}{cccc} p_{\ell 1} & 0 & \cdots & 0 \\ p_{\ell 2} & p_{\ell 1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{\ell d_\ell} & p_{\ell d_\ell-1} & \cdots & p_{\ell 1} \end{array} \right]$$

Displacement generators of the nullspace

Structure of the nullspace basis

$$A \in \mathbb{K}^{m \times n}$$

The nullspace of A is generated by the set

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$$\left\{ N\gamma \in \mathbb{K}^{n \times 1} \mid \gamma \in \mathbb{K}^{\sum_{i=1}^{\ell} t_i} \right\}$$

$$\gamma = \begin{bmatrix} q_{11} \\ \vdots \\ q_{1t_1} \\ \vdots \\ q_{\ell 1} \\ \vdots \\ q_{\ell t_{\ell}} \end{bmatrix} \left\{ \sum_{i=1}^{\ell} t_i \right. \quad N = \begin{bmatrix} N_1 & \cdots & N_{\ell} \end{bmatrix} = \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ p_{12} & p_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{1d_1} & p_{1d_1-1} & \cdots & p_{11} \end{bmatrix} \cdots \begin{bmatrix} p_{\ell 1} & 0 & \cdots & 0 \\ p_{\ell 2} & p_{\ell 1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{\ell d_{\ell}} & p_{\ell d_{\ell}-1} & \cdots & p_{\ell 1} \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Is $\sum_{i=1}^{\ell} t_i \leq n$?

$$n \left\{ \begin{array}{c} \ell \\ \text{ } \end{array} \right. \left\{ \begin{array}{c} \sum_{i=1}^{\ell} t_i \\ \text{ } \end{array} \right. \left\{ \begin{array}{c} G \\ H^T \end{array} \right.$$

Displacement generators of the nullspace

Structure of the nullspace basis

$$A \in \mathbb{K}^{m \times n}$$

The nullspace of A is generated by the set

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Lemma

$$\sum_{i=1}^{\ell} t_i \leq n.$$

Rank

The linear independence of the basis

$$A \in \mathbb{K}^{m \times n}$$

Problem 3:

Rank[Toeplitz-like, α](G, H)

Find the rank of A

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$$\forall i, p_i q_i = 0$$

P is column reduced
and has small degrees



$$\underbrace{\begin{bmatrix} p_1 & p_2 & \cdots & p_\ell \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_P \underbrace{\begin{bmatrix} q_1(x) \\ \vdots \\ q_\ell(x) \end{bmatrix}}_q = 0$$

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Predictable degree property
 $\text{cdeg}_{\text{-shift}}(Pq) = \text{cdeg}_{-t}(q)$

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N has full rank

$$\Rightarrow \text{rank}(A) = n - \sum_{i=1}^{\ell} t_i$$

$$N\gamma = 0$$

$$\forall i, p_i q_i = 0$$

P is column reduced
and has small degrees

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 $\text{cdeg}_{\text{-shift}}(Pq) = \text{cdeg}_{-t}(q)$

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Conclusion

Summary

Structured linear systems, nullspace and rank

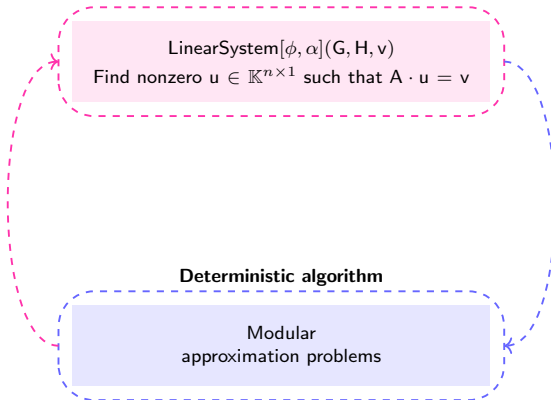
Structured approach

LinearSystem $[\phi, \alpha](G, H, v)$
Find nonzero $u \in \mathbb{K}^{n \times 1}$ such that $A \cdot u = v$

Deterministic algorithm

Modular
approximation problems

Polynomial approach



Summary

Structured linear systems, nullspace and rank

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$\text{Nullspace}[\phi, \alpha](G, H)$

Find $\{z \in \mathbb{K}^{n \times 1} : A \cdot z = 0\}$

$\text{Rank}[\phi, \alpha](G, H)$

Find the rank of A

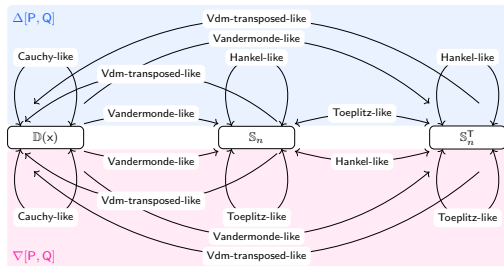
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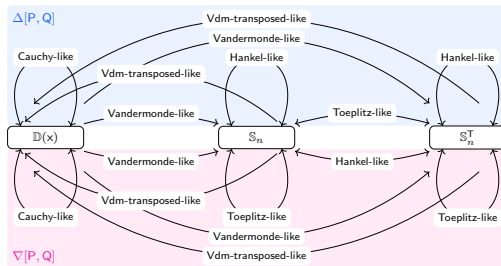
Perspectives

- Extend our result to more general displacement operators,



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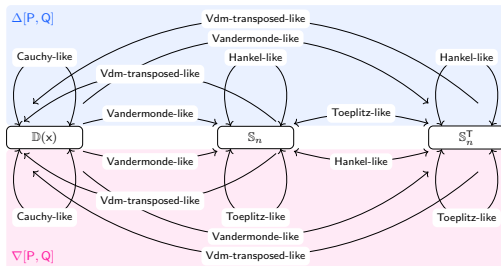
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Perspectives

- Extend our result to more general displacement operators,



- Study cases with multiple nested structures
e.g., $F = [1, p, p^2, \dots, p^{m-1}]$ for some $p \in \mathbb{K}[x]$.

- Compute the inverse (Jeannerod 2025) and determinant,

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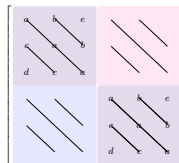
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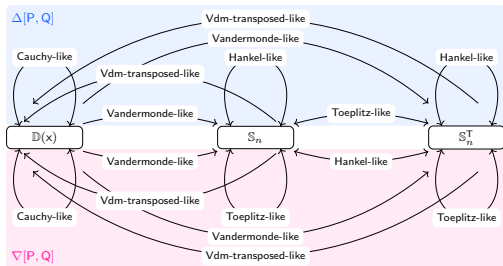
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BTTB

Perspectives

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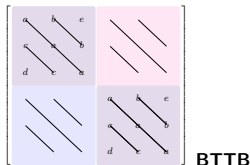
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Thank you!